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THE APPLICATION OF LYAPUNOV FUNCTIONS TO SOME PROBLEMS OF THE ACCEPTABILITY OF APPROXIMATE SOLUTIONS OF DIFFERENTIAL EQUATIONS†

V. N. SKIMEL

Kazan

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The acceptability of approximate solutions of differential equations with respect to some of the variables is considered. The notion of acceptability is defined, generalizing a definition used in [1] in a study of the acceptability of precessional equations of gyroscopic systems. Lyapunov functions are introduced accordingly and used to solve the problem of acceptability. As an application, the possibility of reducing the order of the equations of motion for some mechanical systems is discussed.

THE IDEA of using Lyapunov functions to solve acceptability problems goes back to Chetayev, who pointed out certain features common to such problems and problems of the stability of motion [2].

1. STATEMENT OF THE PROBLEM

Suppose we are given a dynamical system

$$dy/dt = \mathbf{Y}(t, \mathbf{y}, \mathbf{a}), \quad \mathbf{y} \in R^n, \quad \mathbf{a} \in R^r \tag{1.1}$$

where $t \geqslant 0$ is an independent variable (time) and $\mathbf{a}$ is a vector of constant real parameters whose values lie in a given region S ($\mathbf{a} \in S$).

Let us assume that a certain vector-values function

$$\mathbf{u}'(t, \mathbf{a}) = (u_1(t, \mathbf{a}), \ \ldots, \ u_n(t, \mathbf{a})) \tag{1.2}$$

is an approximate solution of system (1.1), and that it is defined and continuous together with its derivative $d\mathbf{u}/dt$ for $t \geqslant t_0$, $\mathbf{a} \in S$ ($\mathbf{u}'$ denotes a column vector).

In order the compare the approximate solution (1.2) with the solutions of system (1.1), we set $\mathbf{y} = \mathbf{u}(t, \mathbf{a}) + \mathbf{x}$. The variables $\mathbf{x}$ will satisfy the following equation

$$d\mathbf{x}/dt = \mathbf{Y}(t, \mathbf{u}(t, \mathbf{a}) + \mathbf{x}, \mathbf{a}) - d\mathbf{u}(t, \mathbf{a})/dt \tag{1.3}$$

Throughout, we shall assume that system (1.3) satisfies the requisite conditions for the existence and uniqueness of its solutions $\mathbf{y} = \mathbf{x}(t, t_0, \mathbf{x}_0, \mathbf{a})$ in the regions under consideration.

Depending on the conditions of the problem, suppose we single out some of the variables $y_1, \ldots, j_m (m < n)$ and the corresponding deviations $x_1, \ldots, x_m$. Define the following regions (parallelepipeds) in the space of the variables $x_1, \ldots, x_m$ by appropriate inequalities

$$\begin{aligned} &\bar{\Pi}_\epsilon = \{ \mathbf{x}: \ |x_\alpha| \leqslant \epsilon_1, \ |x_\beta| \leqslant \epsilon_2 \} \\ &\bar{\Pi}_\delta = \{ \mathbf{x}: \ |x_\alpha| \leqslant \delta_1 < \epsilon_1, \ |x_\beta| \leqslant \delta_2 < \epsilon_2 \}, \qquad (\alpha = 1, \ \ldots, \ m; \ \beta = m + 1, \ \ldots, \ n) \end{aligned} \tag{1.4}$$

The set of boundary points will be denoted by $\bar{\Pi} \backslash \Pi$.

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Definition 1. An approximate solution (1.2) is acceptable with respect to the variables y_α ($\alpha = 1, \ldots, m$) if, for any prescribed numbers $\epsilon_1 > 0$, $\delta_2 > 0$ (the first may be as small as desired and the second large), a parameter value $\mathbf{a}^* \in S$ and numbers $\epsilon_2 > 0$, $\delta_1 > 0$ defining regions (1.4) exist such that, for the solutions of system (1.3), $\mathbf{x}(t, t_0, \mathbf{x}_0, \mathbf{a}^*) \in \Pi_\epsilon$ ($t \geqslant t_0$), whenever $\mathbf{x}_0 \in \bar{\Pi}_\delta$.

2. LYAPUNOV FUNCTION

We will consider real single-valued functions $v = v(t, \mathbf{x}, \mathbf{a})$ which are defined and continuous together with their derivatives $\partial v/\partial t$, $\partial v/\partial x_s$ ($s = 1, \ldots, n$) in a region

$$\Gamma = \{ \mathbf{x}:\ |x_\alpha| \leqslant h,\ |x_\beta| < \infty \}, \qquad t \geqslant t_0,\ \mathbf{a} \in S \tag{2.1}$$

assigning them "property A" [3].

Definition 2. A function $v = v(\mathbf{x}, \mathbf{a})$ possesses property A with respect to variables x_α if, for any $\epsilon_1 > 0$ ($\epsilon_1 < h$), $\delta_2 > 0$ there exist $\mathbf{a}^* \in S$ and numbers $\epsilon_2 > 0$, $\delta_1 > 0$, that depend on the specified ϵ_1 and δ_2, such that the following inequality holds for the regions (1.4) thus defined

$$\inf [v(\mathbf{x}, \mathbf{a}^*):\ \mathbf{x} \in \bar{\Pi}_\epsilon \backslash \Pi_\epsilon] > \sup [v(\mathbf{x}, \mathbf{a}^*):\ \mathbf{x} \in \bar{\Pi}_\delta] \tag{2.2}$$

Definition 3. A function $v = v(t, \mathbf{x}, \mathbf{a})$ possesses property A with respect to variables x_α if there are functions $w(\mathbf{x}, \mathbf{a})$, $W(\mathbf{x}, \mathbf{a})$, defined in the region (2.1), which satisfy the inequality

$$w(\mathbf{x}, \mathbf{a}) \leqslant v(t, \mathbf{x}, \mathbf{a}) \leqslant W(\mathbf{x}, \mathbf{a}) \tag{2.3}$$

and for any $\epsilon_1 > 0$ ($\epsilon_1 < h$), $\delta_2 > 0$ there exist $\mathbf{a}^* \in S$ and numbers $\epsilon_2 > 0$, $\delta_1 > 0$, that depend on the specified ϵ_1 and δ_2, such that the following inequality holds for the regions (1.4) thus defined

$$\inf [w(\mathbf{x}, \mathbf{a}^*):\ \mathbf{x} \in \bar{\Pi}_\epsilon \backslash \Pi_\epsilon] > \sup [W(\mathbf{x}, \mathbf{a}^*):\ \mathbf{x} \in \bar{\Pi}_\delta] \tag{2.4}$$

It is obvious that the functions w and W have property A with respect to x_α in the sense of Definition 2.

Let us consider property A in relation to some functions that are used fairly often in applications.

1. It is obvious that a quadratic form must be positive definite in the region $\mathbf{a} \in S$. Setting $x_\alpha = \xi_\alpha$, $x_\beta = \eta_{\beta - m}$ ($\alpha = 1, \ldots, m$; $\beta = m+1, \ldots, n$; $k = n - m$), we can write a form $v(\mathbf{x}, \mathbf{a}) = \mathbf{x}' D(\mathbf{a}) \mathbf{x}$ $[D(\mathbf{a}) = \|d_{ij}(\mathbf{a})\|\ (i, j = 1, \ldots, n)]$ as

$$v(\mathbf{x}, \mathbf{a}) = \xi' M \xi + 2\xi' N \eta + \eta' L \eta \quad (M = M',\ L = L') \tag{2.5}$$

$$\xi \in R^m, \quad \eta \in R^k, \quad \mathbf{x} = \operatorname{col}(\xi, \eta)$$

The matrices M, N and L are continuous functions of the parameters $\mathbf{a}$ for which v is positive definite.

Let $\|\cdot\|$ denote the Euclidean norm.

Suppose ϵ_1 and δ_1 are specified. Define sets

$$\Gamma_{\epsilon_1} = \{ \mathbf{x}:\ \|\xi\| = \epsilon_1,\ \|\eta\| < \infty \}, \qquad \Gamma_{\delta_2} = \{ \mathbf{x}:\ \|\xi\| = 0,\ \|\eta\| \leqslant \delta_2 \sqrt{k} \} \tag{2.6}$$

The function $v(\mathbf{x}, \mathbf{a})$ of (2.5) has property A with respect to ξ if a value of the parameter $\mathbf{a}^*$ exists such that

$$l_1 = \min [v(\mathbf{x}, \mathbf{a}^*):\ \mathbf{x} \in \Gamma_{\epsilon_1}] > l_2 = \max [v(\mathbf{x}, \mathbf{a}^*):\ \mathbf{x} \in \Gamma_{\delta_2}] \tag{2.7}$$

Indeed, if inequality (2.7) holds, the ellipsoid $v(\mathbf{x}, \mathbf{a}^*) = l_1$ contains the set Γ_{δ_2}; hence, for sufficiently small δ_1, the corresponding region $\bar{\Pi}_\delta$ of (1.4) is also contained in the ellipsoid. The parallelepiped $\bar{\Pi}_\epsilon$ of (1.4) is defined by the same ellipsoid: for sufficiently small ϵ_2, the region $\bar{\Pi}_\epsilon$ contains the ellipsoid. Thus inequality (2.2) is satisfied.

Condition (2.7) may be written in a more-useful form. The stationary values of $v(\mathbf{x}, \mathbf{a})$ (2.5) on the set Γ_{ϵ_1} are defined by expressions $\lambda_i \epsilon_1^2$ ($i = 1, \ldots, m$), where $\lambda_i(\mathbf{a}) > 0$ are the roots of the equation

$$\det(M - NL^{-1}N' - \lambda E) = 0 \tag{2.8}$$

On the other hand, the stationary values of the same form on the set Γ_{δ_2} [see (2.6)] are $\nu_j k\delta_2^2$ ($j = 1, \ldots, k$), where $\nu_j(\mathbf{a}) > 0$ are the roots of the equation

$$\det(L - \nu E) = 0 \tag{2.9}$$

Then condition (2.7) is equivalent to the inequality

$$\lambda_{\min}(\mathbf{a}^*)\epsilon_1^2 > k\nu_{\max}(\mathbf{a}^*)\delta_2^2 \tag{2.10}$$

where $\lambda_{\min}(\mathbf{a}^*)$, $\nu_{\max}(\mathbf{a}^*)$ are the least and greatest roots of Eqs (2.8) and (2.9) when $\mathbf{a} = \mathbf{a}^*$.

For example, for $n = 2$, $m = 1$ inequality (2.10) may be written in terms of the d_{ij}s as follows:

$$(d_{11} - d_{12}^2/d_{22})\epsilon_1^2 > d_{22}\delta_2^2$$

2. Consider a function $v = v(t, \mathbf{x}, \mathbf{a})$ satisfying estimates (2.3)

$$W_1(\mathbf{x}, \mathbf{a}) \leqslant v(t, \mathbf{x}, \mathbf{a}) \leqslant W_2(\mathbf{x}, \mathbf{a})$$

where W_i ($i = 1, 2$) are positive definite quadratic forms in S. Clearly, v possesses property A with respect to x_α if a parameter value $\mathbf{a}^*$ exists such that

$$l_1 = \min[W_1(\mathbf{x}, \mathbf{a}^*):\ \mathbf{x} \in \Gamma_{\epsilon_1}] > l_2 = \max[W_2(\mathbf{x}, \mathbf{a}^*):\ \mathbf{x} \in \Gamma_{\delta_2}] \tag{2.11}$$

Indeed, if this inequality holds, the ellipsoid $W_2(\mathbf{x}, \mathbf{a}^*) = l_1$ will contain the set Γ_{δ_2}; hence, for sufficiency small δ_1, the parallelepiped $\bar{\Pi}_\delta$ of (1.4) will also be in the ellipsoid. The latter, in turn, is a subset of the ellipsoid $W_1(\mathbf{x}, \mathbf{a}^*) = l_1$, that contains it. Thus, we have constructed regions (1.4) satisfying (2.4) for our functions $W_i(\mathbf{x}, \mathbf{a}^*)$.

Inequality (2.11) may be written in the form

$$\lambda_{\min}^{(1)}(a^*)\epsilon_1^2 > k\nu_{\max}^{(2)}(a^*)\delta_2^2, \tag{2.12}$$

where $\lambda_i^{(1)}(\mathbf{a})$, $\nu_j^{(2)}(\mathbf{a})$ are the roots of Eqs (2.8) and (2.9) for the forms W_1 and W_2, respectively.

3. THE METHOD OF LYAPUNOV FUNCTIONS

The application of the method is based on using functions v that have property A, together with their derivatives dv/dt along trajectories of system (1.3)

$$dv/dt = \partial v/\partial t + (\partial v/\partial x)'(\mathbf{Y}(t, \mathbf{u}(t, \mathbf{a}) + \mathbf{x}, \mathbf{a}) - d\mathbf{u}(t, \mathbf{a})/dt) \tag{3.1}$$

The problem of acceptability may be solved effectively by comparing system (1.3) with some system of equations for which there is an available Lyapunov function with the required properties. Equation (1.3) will then be written as follows [2]:

$$d\mathbf{x}/dt = \mathbf{X}(t, \mathbf{x}, \mathbf{a}) + \mathbf{f}(t, \mathbf{x}, \mathbf{a}) \tag{3.2}$$

$$\mathbf{f}(t, \mathbf{x}, \mathbf{a}) = Y(t, \mathbf{u}(t, \mathbf{a}) + \mathbf{x}, \mathbf{a}) - d\mathbf{u}(t, \mathbf{a})/dt - \mathbf{X}(t, \mathbf{x}, \mathbf{a}) \tag{3.3}$$

Let us assume, then, that for the system

$$d\mathbf{x}/dt = \mathbf{X}(t, \mathbf{x}, \mathbf{a}) \tag{3.4}$$

we know of a function $v(t, \mathbf{x}, \mathbf{a})$ with the required properties, whose application to system (3.2) implies that the solution (1.2) is acceptable in the sense of Definition 1.

Suppose that in (3.2) and (3.3) $\mathbf{X}(t, \mathbf{x}, \mathbf{a}) = P(a)\mathbf{x}$, $P(a) = \|p_{ij}(a)\|$ ($i, j = 1, \ldots, n$), $a > 0$.

We shall assume that all the roots of the characteristic equation have negative real parts, and that the functions $f_i(t, \mathbf{x}, \mathbf{a})$ are uniformly bounded with respect to $t \geqslant t_0$ in any bounded closed region of the space $\{x_1, \ldots, x_n\}$. This is the situation in many problems.

Suppose that, given the system $dx/dt = P(a)x$, we have constructed a positive definite quadratic form

$$v(\mathbf{x}, a) = \mathbf{x}'D(a)\mathbf{x} \quad (D = D') \tag{3.5}$$

whose derivative along trajectories of the system is

$$(dv/dt)_* = -2\,\|\mathbf{x}\|^2 \tag{3.6}$$

Theorem 1. Assume that:

1. The form v (3.5) has property A with respect to x_α.
2. For any ϵ_1 and δ_2, a value of the parameter a^* exists such that, in addition to inequality (2.10), it is also true that

$$\epsilon_1(\lambda_{\min}(a^*)/\rho_{\max}(a^*))^{1/2} > n^2NM \tag{3.7}$$

where $\rho_{\max}(a^*)$ is the greatest eigenvalue of the form (3.5), $N = \max|d_{ij}(a^*)|$, $M = \sup|f_i(t, \mathbf{x}, a^*)|$ in the region $\bar{H} = \{\mathbf{x}\colon v(\mathbf{x}, a^*) \leqslant l_1\}$ and $l_1 = \lambda_{\min}(a^*)\epsilon_1^2$.

Then the approximate solution (1.2) is acceptable for system (1.1) with respect to y_α.

Proof. Let ϵ_1 and δ_2 be assigned arbitrarily and let a^* be a value of the parameter satisfying inequalities (2.10) and (3.7). As shown [see (2.7)], since inequality (2.10) holds, the ellipsoid $v(\mathbf{x}, a^*) = l_1$ determines regions (1.4) for which condition (2.2) holds. The radius R of the sphere inscribed in this ellipsoid is found to be $R = \epsilon_1(\lambda_{\min}(a^*)/\rho_{\max}(a^*))^{1/2}$. The derivative of the function (3.5) along trajectories of system (3.2) is determined as follows:

$$dv/dt = -2\,\|\mathbf{x}\|^2 + \sum_{i=1}^{n} f_i(t, \mathbf{x}, a)\frac{\partial v(\mathbf{x}, a)}{\partial x_i} \tag{3.8}$$

Since $|\partial v(\mathbf{x}, a^*)/\partial x_i| \leqslant 2n\|\mathbf{x}\|N$, it follows that the derivative (3.8) satisfies the following estimate in $\bar{H}$

$$dv(\mathbf{x}, a^*)/dt \leqslant -2\|\mathbf{x}\|(\,\|\mathbf{x}\| - n^2NM) \tag{3.9}$$

But condition (3.7) implies that $R > n^2NM$. Hence, by (3.9), $dv(\mathbf{x}, a^*)/dt < 0$ for $\mathbf{x} \in \bar{H}\backslash\bar{I}$, where $\bar{I} = \mathbf{x}\colon \|\mathbf{x}\|^2 \leqslant n^2NM$. This means that no solution $\mathbf{x}(t, t_0, \mathbf{x}_0, a^*)$ of system (3.2) [or equivalently, (1.3)] with initial data $t = t_0$, $\mathbf{x}_0 \in H$, where $H = \mathbf{x}\colon v(\mathbf{x}, a^*) < l_1$ (and, in particular, $\mathbf{x}_0 \in \bar{\Pi}_\delta$) will leave the region at $t > t_0$ (respectively, remaining also in Π_ϵ). Thus the conditions of Definition 1 are satisfied and the solution is acceptable.

We now return to system (3.2), assuming as before that $a > 0$ and the functions $f_i(t, \mathbf{x}, \mathbf{a})$ are uniformly bounded.

Suppose that we have constructed a function $v(t, \mathbf{x}, \mathbf{a})$ for system (3.4) which satisfies estimates partly resembling the conditions for quadratic forms [4]. We are assuming, consequently, that

$$\begin{gathered} W_1(\mathbf{x}, a) \leqslant v(t, \mathbf{x}, a) \leqslant W_2(\mathbf{x}, a) \\ (dv/dt)_* \leqslant -c_1(a)\|\mathbf{x}\|^2, \quad |\partial v/\partial x_i| \leqslant c_2(a)\|\mathbf{x}\|; \quad c_s(a) > 0 \quad (s = 1, 2) \end{gathered} \tag{3.10}$$

where W_s $(s = 1, 2)$ are positive definite quadratic forms, and $(dv/dt)_*$ is the derivative of v along solutions of system (3.4).

Theorem 2. Assume that:

1. The function $v(t, \mathbf{x}, \mathbf{a})$ has property A with respect to x_α.
2. For any numbers ϵ_1 and δ_2 a value of the parameter a^* exists such that, in addition to inequality (2.11), it is also true that

$$\epsilon_1(\lambda_{\min}^{(1)}(a^*)/\rho_{\max}^{(2)}(a^*))^{1/2} > nMc_2(a^*)/c_1(a^*) \tag{3.11}$$

where $\rho_{\max}^{(2)}(a^*)$ is the greatest eigenvalue of the form W_2 of (3.10) and $M = \sup|f_i(t, \mathbf{x}, a^*)|$ in the region $\bar{H}_1 = \{\mathbf{x}\colon W_1(\mathbf{x}, a^*) \leqslant l_1\}$.

Then the approximate solution (1.2) is acceptable for system (1.1) with respect to y_α.

Proof. Let ϵ_1 and δ_2 be arbitrary given numbers and a^* a value of the parameter satisfying inequalities (2.12) and (3.11). As we have shown, when (2.12) is true the ellipsoids $W_s(\mathbf{x}, a^*) = l_1$ ($s = 1, 2$) determine regions (1.4) that satisfy (2.4). The radius of the sphere inscribed in the ellipsoid $W_2(\mathbf{x}, a^*) = l_1$ is $R_2 = (l_1/\rho^2{}_{\max}(a^*))^{1/2}$. The derivative of v along trajectories of system (3.2) is defined as follows:

$$\frac{dv}{dt} = \left(\frac{dv}{dt}\right)_* + \sum_{i=1}^{n} f_i(t, \mathbf{x}, a)\frac{\partial v(t, \mathbf{x}, a)}{\partial x_i} \tag{3.12}$$

Taking note of (3.10), we obtain the following estimate for the derivative in $\bar{H}_1$

$$dv(t, \mathbf{x}, a^*)/dt \leqslant -c_1(a^*)\,\|\mathbf{x}\|\,(\,\|\mathbf{x}\| - nMc_2\,(a^*)/c_1(a^*))$$

But condition (3.11) implies that $R_2 > nMc_2(a^*)/c_1(a^*)$. Consequently, $dv(t, x, a^*)/dt < 0$ if $\mathbf{x} \in \bar{H}_1 \backslash \bar{I}_1$, where

$$\bar{I}_1 = \{\mathbf{x}:\ \|\mathbf{x}\|^2 \leqslant nMc_2(a^*)/c_1(a^*)\}$$

Hence it follows that no solution of system (3.2) with initial data $t = t_0$, $\mathbf{x}_0 \in H_2$, where $H_2 = \{\mathbf{x}: W_2(\mathbf{x}, a^*) < l_1\}$ (and, in particular, $\mathbf{x}_0 \in \bar{\Pi}_\delta$), will leave H_1 (or, respectively, Π_ϵ) at time $t > t_0$. The conditions of Definition 1 are thus satisfied and the approximate solution is acceptable.

Remark 1. If an approximate solution (1.2) of system (1.1) is shown to be acceptable using Theorems 1 and 2, one can show that this same solution is acceptable for the system

$$dy/dt = \mathbf{Y}(t, \mathbf{y}, a) + \epsilon\boldsymbol{\psi}(t, \mathbf{y}, a)$$

which depends on two essential positive parameters a and ϵ. We will limit ourselves to the case in which Eq. (3.2) is

$$d\mathbf{x}/dt = P(a)\mathbf{x} + \mathbf{f}(t, \mathbf{x}, a) + \epsilon\boldsymbol{\psi}(t, \mathbf{u}(t, a) + \mathbf{x}, a) \tag{3.13}$$

In addition to the assumptions of Theorem 1, we assume that

$$|\psi_i(t, \mathbf{u}(t, a^*) + \mathbf{x}, a^*)| < l, \quad \mathbf{x} \in \bar{H}$$

The derivative of the function (3.5) along trajectories of system (3.13) is

$$\frac{dv}{dt} = -2\|\mathbf{x}\|^2 + \sum_{i=1}^{n}(f_i + \epsilon\psi_i)\frac{\partial v}{\partial x_i} \tag{3.14}$$

Then the derivative satisfies the following estimate for a value of the parameter a^* in region H

$$dv(\mathbf{x}, a^*)/dt \leqslant 2\|\mathbf{x}\|\,[\|\mathbf{x}\| - n^2N(M + \epsilon l)]$$

It follows from this inequality that

$$dv(\mathbf{x}, a^*)/dt < 0, \quad \text{if } \|\mathbf{x}\| > n^2N(M + \epsilon l)$$

But we know that for this value of the parameter a^* we have $R > n^2NM$ [or, what is the same thing, (3.7)], while for sufficiently small ϵ also $R > n^2N(M + \epsilon l)$. Thus the assumptions of Theorem 1 are satisfied.

4. APPLICATION OF THE PROBLEM

As an application, we will consider some problems that involve reducing the order of the equations of motion for a mechanical system

$$A\mathbf{q}^{\cdot\cdot} + (B + \Gamma)\mathbf{q}^{\cdot} + C\mathbf{q} = \epsilon\boldsymbol{\varphi}(t, \mathbf{q}, \mathbf{q}^{\cdot}, a) \tag{4.1}$$

where $\mathbf{q}' = (q_1, \ldots, q_m)$ are coordinates, A, B and C are constant symmetric positive definite matrices, Γ is a constant skew-symmetric matrix, $G(a) = B + \Gamma$ is a non-singular matrix, and $a > 0$ and $\epsilon > 0$ are essential parameters.

As an approximation we take the equation

$$G(a)\,\mathbf{p}^{\cdot} + C\mathbf{p} = 0 \qquad (\mathbf{p}' = (p_1, \ldots, p_m)) \tag{4.2}$$

for a gyroscopic system, usually called a precessional system.

We write Eq. (4.2) in normal form

$$\mathbf{p}^{\cdot} = -G^{-1}C\mathbf{p} \tag{4.3}$$

and let $\mathbf{p}(t, a)$ be a solution of this equation at t_0 for fixed $\mathbf{p}_0$.

Clearly, the equilibrium position $\mathbf{p} = 0$ of system (4.3) is asymptotically stable, as is the equilibrium position $\mathbf{q} = 0$ of system (4.1) when $\varphi \equiv 0$.

Let us consider the acceptability of this approximate solution for Eq. (4.1) with respect to the variable $\mathbf{q}$. Putting $\mathbf{z} = \mathbf{q} - \mathbf{p}(t, a)$, substituting into Eq. (4.1) and noting that $\mathbf{p}(t, a)$ is a solution of Eq. (4.2), we see that for the variable $\mathbf{z}$

$$\mathbf{z}^{\cdot\cdot} + A^{-1}G\mathbf{z}^{\cdot} + A^{-1}C\mathbf{z} = -\mathbf{p}^{\cdot\cdot}(t, a) + \epsilon A^{-1}\varphi(t, \mathbf{p}(t, a) + \mathbf{z}, \mathbf{p}^{\cdot}(t, a) + \mathbf{z}^{\cdot}, a) \tag{4.4}$$

In variables $\mathbf{z}$ and $\mathbf{z}^{\cdot}$, Eq. (4.4) may be associated with a system of type (3.13) if we put $\mathbf{x} = \mathrm{col}(z, z^{\cdot})$, $\mathbf{x} \in R_n$ $(n = 2m)$, and also

$$P(a) = \begin{Vmatrix} 0 & E \\ -A^{-1}C & -A^{-1}G \end{Vmatrix}, \qquad \mathbf{f} = \mathrm{col}(\mathbf{0}, -\mathbf{p}^{\cdot\cdot}(t, a)), \qquad \psi = \mathrm{col}(\mathbf{0}, A^{-1}\varphi) \tag{4.5}$$

Thus, in the notation of (1.4) we must put

$$x_\alpha = z_\alpha, \quad x_\beta = z_\alpha^{\cdot} \quad (\alpha = 1, \ldots, m;\ \beta = m + \alpha)$$

To solve the acceptability problem, we must construct a quadratic form (3.5) for the homogeneous linear system with matrix $P(a)$ as in (4.5) and check the assumptions of Theorem 1, with due attention to Remark 1. Expression (3.8) for the derivative becomes

$$dv/dt = -2\,\|\mathbf{x}\|^2 - 2\sum_{\alpha=1}^{m}\sum_{i=1}^{n} p_\alpha^{\cdot\cdot}(t, a)\, d_{m+\alpha,\, i}\, x_i \tag{4.6}$$

Accordingly, we must put

$$N = \max|d_{m+\alpha,\, i}(a^*)|, \qquad |p_\alpha^{\cdot\cdot}(t, a^*)| < M \quad (\alpha = 1, \ldots, m;\ i = 1, \ldots, n)$$

in inequality (3.9). By (4.3), we get

$$\mathbf{p}^{\cdot\cdot}(t, a) = K^2(a)\,\mathbf{p}(t, a), \qquad K(a) = G^{-1}C \tag{4.7}$$

Remark 2. If B is a positive or negative semidefinite diagonal matrix, then the matrix $G = B + \Gamma$ is non-singular. If at the same time B is positive definite, then $\det\|B + \Gamma\| > 0$ [5].

Example 1. The acceptability of the precessional equations of a gyroscope. Suppose that Eqs (4.1) are

$$\begin{aligned} q_1^{\cdot\cdot} + bq_1^{\cdot} - aq_2^{\cdot} + c_1 q_1 &= \epsilon\varphi_1(t, \mathbf{q}, \mathbf{q}^{\cdot}, a) \\ q_2^{\cdot\cdot} + bq_2^{\cdot} + aq_1^{\cdot} + c_2 q_2 &= \epsilon\varphi_2(t, \mathbf{q}, \mathbf{q}^{\cdot}, a) \end{aligned} \tag{4.8}$$

where a, b, c_1, c_2 and ϵ are positive constants; and a and ϵ are essential parameters.

The approximate equations (4.3) may be written in the form

$$\begin{aligned} &p_1^{\cdot} = -\mu(\mu bc_1 p_1 + c_2 p_2)/\Delta(\mu), \qquad p_2^{\cdot} = -\mu(\mu bc_2 p_2 - c_1 p_1)/\Delta(\mu) \\ &\mu = a^{-1}, \qquad \Delta(\mu) = 1 + \mu^2 b^2 \end{aligned} \tag{4.9}$$

The solution of Eqs (4.9) for fixed $\mathbf{p}_0$ is bounded

$$|p_{1,2}(t, \mu)| < h_1 \quad (t \geqslant 0)$$

By (4.7), we obtain

$$p_1^{\cdot\cdot} = \mu^2 [-c_1(c_2 - \mu^2 b^2 c_1) p_1(t, \mu) + \mu c_2 b(c_1 + c_2) p_2(t, \mu)]/\Delta^2(\mu)$$
$$p_2^{\cdot\cdot} = \mu^2 [-\mu c_1 b(c_1 + c_2) p_1(t, \mu) - c_2(c_1 - \mu^2 b^2 c_2) p_2(t, \mu)]/\Delta^2(\mu)$$

from which we obtain the estimate

$$|p_{1,2}^{\cdot\cdot}(t, \mu)| < \mu^2 h_1 (c_1 c_2 + O(\mu^3)) \tag{4.10}$$

In variables $\mathbf{x}$, we obtain the following system of equations for (3.2)

$$\begin{aligned} &dx_1/dt = x_3, \quad dx_2/dt = x_4 \\ &dx_3/dt = -c_1 x_1 - bx_3 + \mu^{-1} x_4 - p_1^{\cdot\cdot}(t, \mu) \\ &dx_4/dt = -c_2 x_2 - \mu^{-1} x_3 - bx_4 - p_2^{\cdot\cdot}(t, \mu) \end{aligned} \tag{4.11}$$

The coefficients of the form v of (3.5) are defined as follows:

$$\begin{aligned} &d_{11} = \frac{[\mu^{-2}c_1 + b^2 c_2 + (1 + c_1) c_1 c_2]}{bc_1 c_2}, \quad d_{12} = \frac{-\mu^{-1}(c_2 - c_1)}{c_1 c_2}, \\ &d_{13} = \frac{1}{c_1}, \quad d_{14} = \frac{\mu^{-1}}{bc_2}, \quad d_{22} = \frac{[\mu^{-2}c_2 + b^2 c_1 + (1 + c_2) c_1 c_2]}{bc_1 c_2} \\ &d_{23} = \frac{-\mu^{-1}}{bc_1}, \quad d_{24} = \frac{1}{c_2}, \quad d_{33} = \frac{1 + c_1}{bc_1}, \quad d_{34} = 0, \quad d_{44} = \frac{1 + c_2}{bc_2} \end{aligned} \tag{4.12}$$

We now proceed to inequalities (2.10) and (3.7). Using expression (2.5) for the form v, assuming, to fix our ideas, that $c_2 > c_1$ and using Eq. (2.8), we find that if μ is small enough, then $\lambda_{\min}(\mu) \cong -2\mu^{-2} b(1 + c_2)$. The roots of Eq. (2.9) are independent of μ. Thus, for any given ϵ_1 and δ_2, inequality (2.10) will be true, provided we take μ^* small enough.

If μ is sufficiently small, it follows from (4.10), (4.11) and (4.6) that inequality (3.7) will hold with $N = \mu^{-1}/(bc_2)$, $M \cong \mu^2 h_1 c_1 c_2$. It is immediately seen from the equation $|D(\mu) - \rho E| = 0$ that the quantity $\rho_{\max}(\mu)$ is of the same order of magnitude in μ as $\lambda_{\min}(\mu)$. Inequality (3.7) is thus true for sufficiently small μ^*. It follows from Remark 1 that the solutions of Eqs (4.9) are acceptable for system (4.8) with respect to the coordinates.

Example 2. Consider the system

$$q_\alpha^{\cdot\cdot} + ab_\alpha q_\alpha^{\cdot} + c_\alpha q_\alpha = \epsilon \varphi_\alpha(t, \mathbf{q}, \mathbf{q}^{\cdot}, a) \quad (\alpha = 1, \ldots, m) \tag{4.13}$$

in which b_α and c_α are positive constants, and $a > 0$ and $\epsilon > 0$ are essential parameters.

The approximate equations will be

$$p_\alpha^{\cdot} = -\mu \gamma_\alpha p_\alpha \quad (\mu = a^{-1}, \ \gamma_\alpha = c_\alpha / b_\alpha) \tag{4.14}$$

Their solution is $p_\alpha(t, \mu) = p_{\alpha 0} \exp\{-\mu \gamma_\alpha t\}$, so that $p_\alpha^{\cdot\cdot}(t, \mu) = \mu^2 \gamma_\alpha^2 p_\alpha(t, \mu)$. Since $|p_\alpha(t, \mu)| < h_1$ $(t \geqslant 0)$, it follows that $|p_\alpha^{\cdot\cdot}(t, \mu)| < \mu^2 h_2$, where h_1 and h_2 are suitably defined constants.

Putting $x_\alpha = q_\alpha - p_\alpha(t, \mu)$, $x_\beta = q^{\cdot}_\alpha - p^{\cdot}_\alpha(t, \mu)$ as before, where $\beta = m + \alpha$, we obtain a system of equations

$$dx_\alpha/dt = x_\beta, \quad dx_\beta/dt = -c_\alpha x_\alpha - \mu^{-1} b_\alpha x_\beta - p_\alpha^{\cdot\cdot}(t, \mu) + \epsilon \varphi_\alpha(t, \mathbf{x}, \mu) \tag{4.15}$$

which splits at $\epsilon = 0$ into m pairs of independent systems of equations of the type (3.2) in the variables x_α and x_β.

For each pair of equations, we define a form (3.5)

$$v_\alpha = d_{\alpha\alpha} x_\alpha^2 + 2 d_{\alpha\beta} x_\alpha x_\beta + d_{\beta\beta} x_\beta^2 \tag{4.16}$$

whose derivative (3.6) is $(dv_\alpha/dt)_* = -2(x_\alpha^2 + x_\beta^2)$, and the coefficients are defined by

$$d_{\alpha\alpha} = \frac{\mu[\mu^{-2} b_\alpha^2 + c_\alpha(1 + c_\alpha)]}{b_\alpha c_\alpha}, \quad d_{\alpha\beta} = \frac{1}{c_\alpha}, \quad d_{\beta\beta} = \frac{\mu(1 + c_\alpha)}{b_\alpha c_\alpha} \tag{4.17}$$

For system (4.15) we use the form (4.16) $v = v_1 + \ldots + v_m$. Its derivative along trajectories of the system is, by (3.14) and (4.6)

$$\frac{dv}{dt} = -2\|\mathbf{x}\|^2 - 2\sum_{\alpha=1}^{m} [p_\alpha^{\cdot\cdot}(t, \mu) + \epsilon\varphi_\alpha(t, \mathbf{x}, \mu)](d_{\alpha\beta}x_\alpha + d_{\beta\beta}x_\beta)$$

We now look at inequalities (2.10) and (3.7). Clearly, the roots of Eqs (2.8) and (2.9) are $\lambda_\alpha(\mu) = d_{\alpha\alpha} - d_{\alpha\beta}^2/d_{\beta\beta}$, $\nu_\alpha(\mu) = d_{\beta\beta}$. Considering expressions (4.17), we conclude that inequality (2.10) will hold for sufficiently small values of μ. The truth of inequality (3.7) for sufficiently small μ is no less obvious. Indeed, as $\mu \to 0$ the right-hand side of (3.7) decreases without limit. The left-hand side has a non-zero limit. We note that the eigenvalues $\rho(\mu)$ of the form v are determined directly from the equations

$$(d_{\alpha\alpha} - \rho)(d_{\beta\beta} - \rho) - d_{\alpha\beta}^2 = 0 \quad (\alpha = 1, \ldots, m;\ \beta = m + \alpha)$$

Choosing a sufficiently small value of μ^* to satisfy conditions (2.10) and (3.7), and using Remark 1, we determine the value of the second essential parameter ϵ. This shows that the solution of (4.14) is acceptable for system (4.13).

In conclusion, we point out that the equations of motion of systems whose matrices of dissipative and conservative forces have the form

$$B = a\|b_\alpha\delta_{\alpha j}\|_1^m + \epsilon\|b_{\alpha j}\|_1^m, \qquad C = \|c_\alpha\delta_{\alpha j}\|_1^m + \epsilon\|c_{\alpha j}\|_1^m$$

can be reduced to the form of (4.13).

In the case of system (4.13), estimates of the acceptability of a solution of (4.14) will also determine whether the system can be decoupled.

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